A&W

Exercise Session 6

Probability I

Nil Ozer

A&W Overview

A&W

Connectivity
- Articulation Points
- Bridges
- Block - Decomposition
- Menger's Theorem

Cycles
- Eulerian Cycle
- Hamiltonian Cycle
- TSP

Matchings
- Definition
- Algorithms
- Hall's Theorem

Colorings
- Definition
- Algorithm
- Brooks's Theorem

Wahrscheinlichkeit
- Grundbegriffe und Notationen
- Bedingte Wahrscheinlichkeiten
- Unabhängigkeiten
- Zufallsvariablen
- Wichtige Diskrete Verteilungen
- Abschätzen von Wahrscheinlichkeiten

Randomized Algorithms
- Las-Vegas
- Monte-Carlo
- Primality Test
- Target - Shooting
- Finding Duplicates
- Longest Path Problem

Graph Algorithms

Flow
- Definition
- Maxflow - Mincut
- Ford - Fulkerson
- Matching w. Flow
- Edge-disjoint paths w. Flow

Minimum Cut
- Definition
- Cut(G) Algorithm
- Bootstrapping

Geometric Algorithms

Convex Hull
- Definition
- Jarvis Wrap
- Local optimization

Smallest Enclosing Circle
- Definition
- First Algorithm
- Final Algorithm

Outline

- Minitest 3
- T2 Discussion
- Cycles + TSP Kahoot
- Formelsammlung
- Probability Theory

Minitest 3

T2
Feedback + Discussion

- Watch out for the comments !
- Keep up the good work ! 👏🏻👏🏻👏🏻
- Questions, issues … Let me know !
  - After the weekend

Cycles + TSP Kahoot

Let's take a break

A&W 😊🧑‍💻

WhatsApp group

Probability Theory

Formelsammlung

Probability Theory

Importance for CS

- A&W
- NumCS
- WuS
- IML
- InfoSec
- Information Retrieval
- Data Science
- Anything ML related
- Anything AI related

Probability Theory

Importance for CS

LEARN IT ONCE !

Probability Theory

Basic Terms of Discrete Probability Space

- Elementary Event ω_i (Elementarereignis) : A single outcome of the experiment
- Sample Space Ω (Ergebnismenge) : The set of all possible outcomes $\Omega = \{\omega_1, \omega_2, ...\}$
- Elementary Probability $Pr[\omega_i]$ (Elementarwahrcheinlichkeit) : Probability of an elementary event

- Event E (Ereignis) : A subset of the sample space ($E \subseteq \Omega$)
- Complementary event $\overline{E}$ (Komplementärereignis) : All outcomes not in E $\overline{E} := \Omega \setminus E$

Probability Theory

Pr[ ] Properties

- $0 \leq Pr[\omega_i] \leq 1$
- $\sum_{\omega \in \Omega} \Pr[\omega] = 1$
- Probability of an event E : $Pr[E] := \sum_{\omega \in E} \Pr[\omega]$

Probability Theory

Pr[ ] Properties

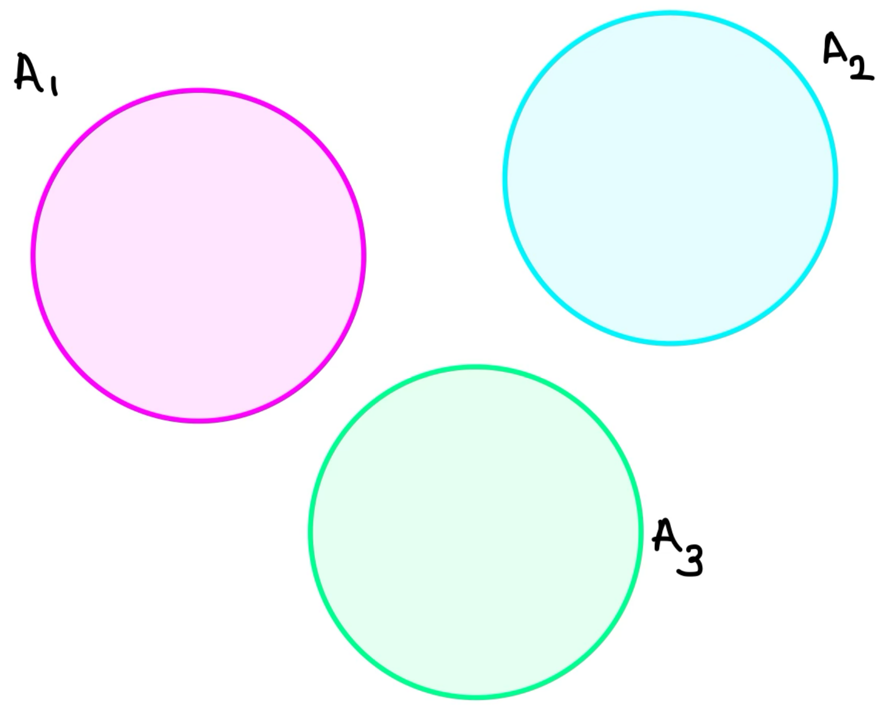
- For all events $A, B, A_1, A_2, \ldots$
  - $Pr[\varnothing] = 0\,, Pr[\Omega] = 1$
  - $0 \le Pr[A] \le 1$
  - $Pr[\overline{A}] = 1 - Pr[A]$
  - $A \subseteq B \implies Pr[A] \le Pr[B]$

- $0 \le Pr[\omega_i] \le 1$
- $\sum_{\omega \in \Omega} \Pr[\omega] = 1$
- Probability of an event E : $Pr[E] := \sum_{\omega \in E} \Pr[\omega]$

Examples I

Probability Theory

Addition Rule



Events $A_1, \ldots, A_n$ are pairwise disjoint $\implies$ $\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i]$

Infinite set of events $A_1, A_2 \ldots$ are pairwise disjoint $\implies$ $\Pr\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \Pr[A_i]$

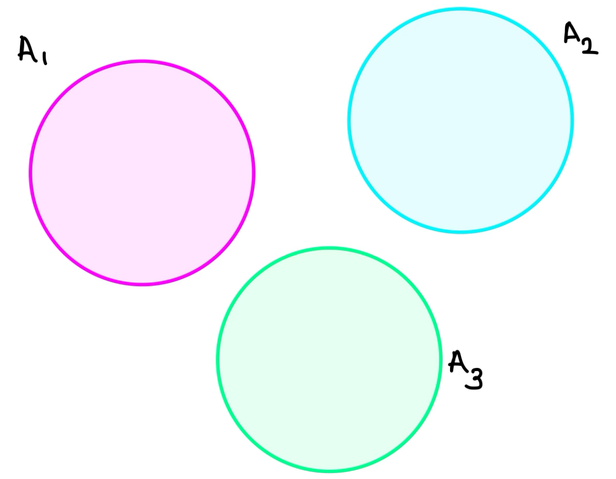
pairwise disjoint : For all pairs A_i, A_j with $i \neq j$: $A_i \cap A_j = \emptyset$

Probability Theory
Boolean Inequality

Addition Rule

Events $A_1, \ldots, A_n$ are pairwise disjoint

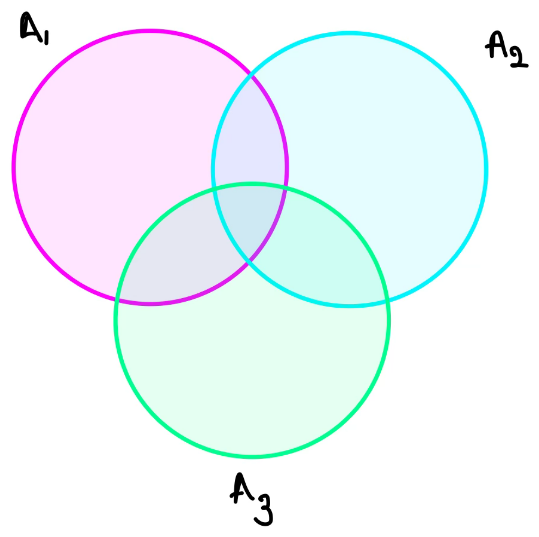
$$\implies \Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i]$$



Boolean Inequality

For arbitrary events $A_1, \ldots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$$



Probability Theory

Inclusion-Exclusion Principle (Siebformel)

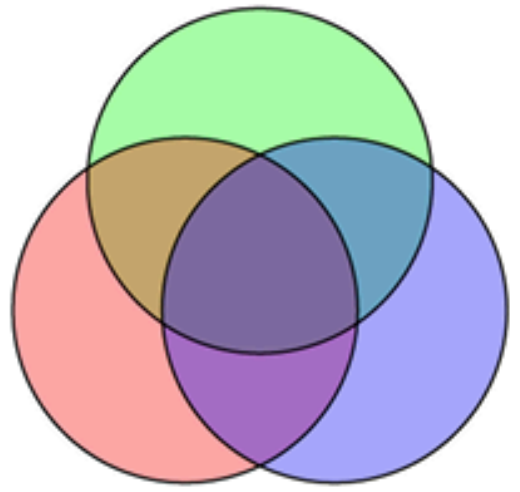
For events $A_1, \ldots, A_n$

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{1 \le i_1 < i_2 \le n} \Pr[A_{i_1} \cap A_{i_2}] + \cdots$$

$$+(-1)^{l+1} \sum_{1 \le i_1 < \cdots < i_l \le n} \Pr[A_{i_1} \cap \cdots \cap A_{i_l}] + \cdots$$

$$+(-1)^{n+1} \Pr[A_1 \cap \cdots \cap A_n] \,.$$

n=3:



Pr[A ∪ B ∪ C]

= Pr[A] + Pr[B] + Pr[C]

- Pr[A∩B] - Pr[A∩C] - Pr[B∩C] + Pr[A∩B∩C]

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Probability Theory

Inclusion-Exclusion Principle (Siebformel)

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{1 \le i_1 < i_2 \le n} \Pr[A_{i_1} \cap A_{i_2}] + \cdots$$
$$+ (-1)^{l+1} \sum_{1 \le i_1 < \cdots < i_l \le n} \Pr[A_{i_1} \cap \cdots \cap A_{i_l}] + \cdots$$
$$+ (-1)^{n+1} \Pr[A_1 \cap \cdots \cap A_n] .$$

To find the cardinality of the union of n sets:

1. Include the cardinalities of the sets.
2. Exclude the cardinalities of the pairwise intersections.
3. Include the cardinalities of the triple-wise intersections.
4. Exclude the cardinalities of the quadruple-wise intersections.
5. Include the cardinalities of the quintuple-wise intersections.
6. Continue, until the cardinality of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).

Probability Theory

Inclusion-Exclusion Principle (Siebformel)

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{1\le i_1<i_2\le n} \Pr[A_{i_1}\cap A_{i_2}] + \cdots$$
$$+(-1)^{l+1} \sum_{1\le i_1<\cdots<i_l\le n} \Pr[A_{i_1}\cap\cdots\cap A_{i_l}] + \cdots$$
$$+(-1)^{n+1} \Pr[A_1\cap\cdots\cap A_n]\,.$$

To find the probability of the union of n sets:

1. Include the probability of the sets.
2. Exclude the probability of the pairwise intersections.
3. Include the probability of the triple-wise intersections.
4. Exclude the probability of the quadruple-wise intersections.
5. Include the probability of the quintuple-wise intersections.
6. Continue, until the probability of the n-tuple-wise intersection is included (if n is odd) or excluded (n even).

Examples II

Probability Theory
Laplace-Space

- Laplace-Space :
  - A finite probability space where all outcomes/elementary events have the same probability

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$$

$$Pr[\omega_i] = \frac{1}{n} \quad \text{for all } i = 1, \ldots, n$$

$$Pr[A] = \frac{|A|}{|\Omega|}$$

Probability Theory
Combinatorics

different ways of drawing k elements from n options

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Probability Theory

Combinatorics

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Q : Create a 3-digit password using digits 0-9.

Repetition : allowed

Order : matters

Formula : n^k

A :

- n = 10
  - There are 10 options
  - digits 0-9
- k = 3
  - We draw 3 elements in order
  - password length is 3

$$10^3 = 10 \cdot 10 \cdot 10 = 1000$$

Probability Theory
Combinatorics

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Q : Choose 3 students to present one after another from a group of 5

Repetition : not allowed

Order : matters

A :

- n = 5
  - There are 5 options
  - student group of 5
- k = 3
  - We draw 3 elements in order
  - 3 students will present

Formula : $n^{\underline{k}}$

$$n^{\underline{k}} := \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}_{k \text{ Faktoren}}$$

$$5^{\underline{3}} = 5 \cdot 4 \cdot 3 = 60$$

Probability Theory
Combinatorics

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Q : You have 6 friends. Choose 2 of them to go on a trip.

Repetition : not allowed

Order : doesn't matter

Formula : $\binom{n}{k}$

A :
- n = 6
  - There are 6 options
  - you have 6 friends
- k = 2
  - We draw 2 elements without order
  - 2 friends will come

$$\binom{6}{2} = \left(\frac{6!}{2! \cdot (6-2)!}\right) = \left(\frac{6!}{2! \cdot 4!}\right) = 15$$

$$\binom{6}{2} = \left(\frac{6 \cdot 5}{1 \cdot 2}\right) = 15$$

Probability Theory

Combinatorics

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Q : Buy 4 scoops of ice cream from 3 flavors (vanilla, chocolate, strawberry)

Repetition : allowed

Order : doesn't matter

A :

- n = 3
  - There are 3 options
  - 3 flavors
- k = 4
  - We draw 4 elements without order
  - 4 scoops of ice cream

Formula :

$$\binom{n+k-1}{k}$$

$$\binom{3+4-1}{4} = \binom{6}{4} = 15$$

Do you notice something ?

Probability Theory
Combinatorics

	order matters	order does not matter
repetition allowed	n^k	$\binom{n+k-1}{k}$
repetition not allowed	$n^{\underline{k}}$	$\binom{n}{k}$

Q : Buy 4 scoops of ice cream from 3 flavors (vanilla, chocolate, strawberry)

Repetition : allowed

Order : doesn't matter

Formula :

$$\binom{n+k-1}{k}$$

A :

- n = 3
  - There are 3 options
  - 3 flavors
- k = 4
  - We draw 4 elements without order
  - 4 scoops of ice cream

Do you notice something ?

$$\binom{3+4-1}{4} = \binom{6}{4} = 15 = \binom{6}{2}$$

Probability Theory

Conditional Probability

the probability that event A will occur if we already know that event B has occurred

- conditional probability :
  - Let A and B be arbitrary events with $Pr[B] > 0$. The conditional probability $Pr[A \mid B]$ of A given B is

$$Pr[A \mid B] := \frac{Pr[A \cap B]}{Pr[B]}$$

$$Pr[A \cap B] = Pr[A \mid B] \cdot Pr[B]$$

$$Pr[A \cap B] = Pr[B \mid A] \cdot Pr[A]$$

Probability Theory

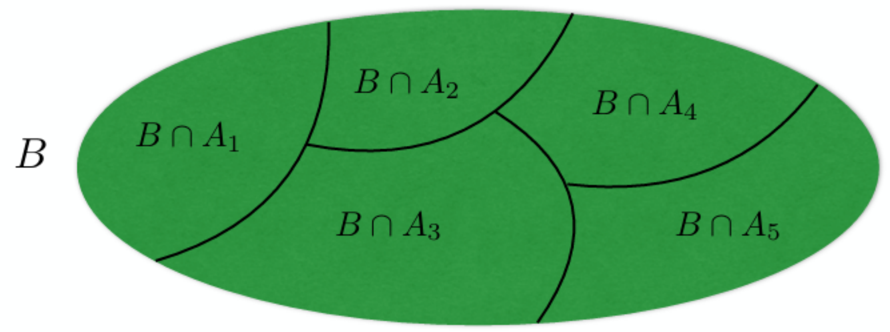
Conditional Probability - Theorems

Bedingte Wahrscheinlichkeiten

- **Definition:** Ist $\Pr[B] > 0$, so ist $\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}$.
- **Multiplikationssatz:** Ist $\Pr[A_1 \cap \ldots \cap A_n] > 0$, so ist

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \ldots \cdot \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

- **Satz von der totalen Wahrscheinlichkeit:**
  Ist $\Omega = A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n] > 0$, so gilt $\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$.



- **Satz von Bayes:**
  Ist $B \subseteq A_1 \uplus \cdots \uplus A_n$ mit $\Pr[A_1], \ldots, \Pr[A_n], \Pr[B] > 0$, so gilt

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

Examples III

Probability Theory

Conditional Independence

- conditional independence :
  - Event A and B are independent , if

$$Pr[A \cap B] = Pr[A] + Pr[B]$$

  - Events A ,B and C are independent , if

$$\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$$

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$$
$$\Pr[A \cap C] = \Pr[A] \cdot \Pr[C]$$
$$\Pr[B \cap C] = \Pr[B] \cdot \Pr[C]$$

Minitest 3 - Discussion

Questions

Feedbacks , Recommendations

Nil Ozer