Quiz 1

1) $n^6 + 10n^5 + 20n^2 = O(n^6)$

$$\lim_{n\to\infty} \frac{n^6 + 10n^5 + 20n^2}{n^6} = \lim_{n\to\infty} \frac{n^6\left(1 + \frac{10n^5}{n^6} + \frac{20n^2}{n^6}\right)}{n^6} = \lim_{n\to\infty} 1 + \underbrace{\frac{10}{n}}_{\to 0} + \underbrace{\frac{20}{n^4}}_{\to 0} = 1 \in \mathbb{R}^+$$

2) $n^4 - 5n^2 \leq O(\frac{n^4}{\log n})$.

$$\lim_{n\to\infty} \frac{n^4 - 5n^2}{\frac{n^4}{\log n}} = \lim_{n\to\infty} \frac{(n^4 - 5n^2)\cdot \log n}{n^4} = \lim_{n\to\infty} \left(1 - \frac{5}{n^2}\right)\cdot \log n = \lim_{n\to\infty} \log n - \underbrace{\frac{5\log n}{n^2}}_{\to 0} = \infty$$

3) Which of these statements is correct?

1. $e^{3\log(n)} \leq O(n)$
2. $e^{\log(3)+\log(n)} \leq O(n)$

1. $e^{3\log(n)} = e^{\log n^3} = n^3 \not\leq O(n)$ 1 is wrong
2. $e^{\log(3)+\log(n)} = e^{\log 3n} = 3n \leq O(n)$ 2 is correct

4) Let $f : \mathbb{N} \to \mathbb{R}$, $g : \mathbb{N} \to \mathbb{R}$ and $h : \mathbb{N} \to \mathbb{R}$ be three functions.
Assume that $f(n) \leq O(n \cdot g(n))$ and $g(n) \leq O(n^2 \cdot h(n))$.
Then it must be true that $f(n) \leq O(n^3 h(n))$.

$f(n) \leq O(n\cdot g(n)) \Rightarrow \exists C_1 > 0$ s.t. $f(n) \leq C_1 \cdot n \cdot g(n)$ for all $n \in \mathbb{N}$

$g(n) \leq O(n^2 \cdot h(n)) \Rightarrow \exists C_2 > 0$ s.t. $g(n) \leq C_2 \cdot n^2 \cdot h(n)$ for all $n \in \mathbb{N}$

$\Rightarrow f(n) \leq C_1 \cdot C_2 \cdot n^3 h(n)$ for all $n \in \mathbb{N}$ ($\leq$ is transitive)

$\Rightarrow f(n) \leq C_3 \cdot n^3 h(n) \Rightarrow f(n) \leq O(n^3 h(n))$

$(C_3 = C_1 \cdot C_2)$

5) We want to prove that $2^{2n} \leq (2n)!$ for all integers $n \geq 2$ using induction.
Which of the following is a correct *base case* for the inductive argument?

a. $2^2 = 4 \leq 24 = 4! = (2\cdot 2)!$?

b. $2^{2\cdot 2} = 2^4 = 16 \leq 24 = 4! = (2\cdot 2)!$

c. For some k, assume that $2^{2k} \leq (2k)!$ I.H.

d. $2^{2\cdot 3} = 2^6 = 64 \leq 720 = 6! = (2\cdot 3)!$ $(n \geq 3)$

what is used ?

Theorem 1. *Let N be an infinite subset of $\mathbb{N}$ and $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, then $f \leq O(g)$ and $g \not\leq O(f)$.*
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, then $f \leq O(g)$ and $g \leq O(f)$.*
- *If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, then $f \not\leq O(g)$ and $g \leq O(f)$.*

$\boxed{b^x = (e^{\ln b})^x = e^{(\ln b)\cdot x}.}$

$$\log_a(bc) = \log_a(b) + \log_a(c)$$
$$\log_a(b^c) = c\log_a(b)$$

Definition 1 (O-Notation). For $f : N \to \mathbb{R}^+$,

$$O(f) := \{g : N \to \mathbb{R}^+ \mid \exists C > 0\, \forall n \in N\; g(n) \leq C \cdot f(n)\}.$$

Induction: base case explanation from last week