Graph Definitions - Short Proofs

In the following, let $G = (V, E)$ be a graph, $n = |V|$ and $m = |E|$.

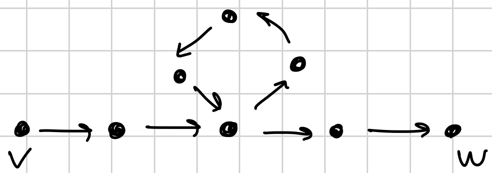
(a) Let $v \neq w \in V$. if there is a walk with endpoints v and w, then there is a path with endpoints v and w. T/F ? Proof ?

True

$\exists$ walk W with endpoints v and w

If all vertices in W are distinct, W is already a path

If W contains repeated vertices, remove the cycles (to obtain a path from v to w)



If there exists a cycle $v_{i-1}, v_i, v_{i+1}, \dots, v_i$, v_j, v_{j+1}

Remove the vertices $v_{i+1} \dots v_i$

$v_{i-1}, v_i, v_j, v_{j+1} \dots$

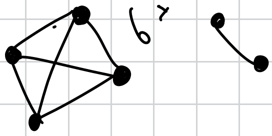
Repeat this for every remaining repeated vertices

After removing all cycles, remaining walk will be a path (has distinct vertices + still connects v to w)

(b) Every graph with $m \geq n$ is connected.

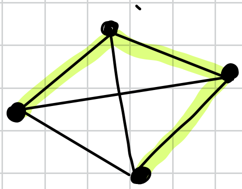
False

$6+1=7$



(c) If G contains a Hamiltonian path, then G contains a Eulerian walk.

False



But every vertex has odd degree

$\Longleftrightarrow$

no eulerion walk

(at most 2 vertices could have odd degree, start and end)

(d) If every vertex of a non-empty graph G has degree at least 2, then G contains a cycle.

True

Proof by using properties of a tree

Assume that G does not contain a cycle

$\Rightarrow$ All of it's connected components must be a tree

$\Rightarrow |E| \leq |V|-1$

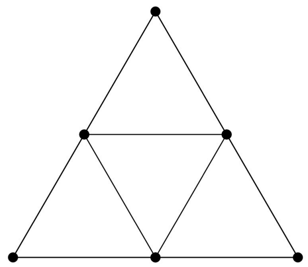
But each vertex has degree at least 2 $\rightarrow$ $\frac{2 \cdot |V|}{2} = |V|$ edges

Contradiction!

G has to contain a cycle!

(e) Suppose in a graph G every pair of vertices v, w has a common neighbour (i.e., for all distinct vertices v, w, there is a vertex x such that $\{v, x\}$ and $\{w, x\}$ are both edges). Then there exists a vertex p in G which is a neighbour of every other vertex in G (i.e., p has degree $n - 1$).

False



(f) Let G be a connected graph with at least 3 vertices. Suppose there exists a vertex v_{cut} in G so that after deleting v_{cut}, G is no longer connected. Then G does not have a Hamiltonian cycle. (Deleting a vertex v means that we remove v and any edge containing v from the graph).

True

Indirect proof

Assume G has a hamiltonian cycle

Resulting G after removing v_{cut} would still have a hamiltonian path.

It's still connected!