Loop Counting Examples

Exercise 3.3 *Counting function calls in loops* (**1 point**).

For each of the following code snippets, compute the number of calls to f as a function of $n \in \mathbb{N}$. Provide **both** the exact number of calls and a maximally simplified asymptotic bound in Θ notation.

Algorithm 1

(a)
```
i ← 0
while i ≤ n do
    f()
    f()
    i ← i + 1
j ← 0
while j ≤ 2n do
    f()
    j ← j + 1
```

$$\sum_{i=0}^{n} 2 + \sum_{j=0}^{2n} 1 = (n-0+1)\cdot 2 + (2n-0+1)\cdot 1$$

$$= 2n+2+2n+1 = 4n+3 = \Theta(n)$$

Algorithm 2

(b)
```
i ← 1
while i ≤ n do
    j ← 1
    while j ≤ i^3 do
        f()
        j ← j + 1
    i ← i + 1
```

$$\sum_{i=1}^{n} \sum_{j=1}^{i^3} 1 = \sum_{i=1}^{n} (i^3-1+1)\cdot 1 = \sum_{i=1}^{n} i^3 \overset{!}{=} \frac{n^2\cdot(n+1)^2}{4} = \Theta(n^4)$$

for the mini cheat sheet:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

FS 23

Theory Task T2. / 15 P

In this part, you should **justify your answers briefly**.

/ 4 P

a) *Counting iterations:* For the following code snippets, derive an asymptotic bound for the number of times f is called. Simplify the expression as much as possible and state it in Θ-notation as concisely as possible.

i) Snippet 1:

Algorithm 1

```
for i = 1,...,n do
    for j = 1,...,i^2 do
        f()
    f()
```

$$\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{i^2} 1\right)+1\right) = \sum_{i=1}^{n} i^2+1 = \sum_{i=1}^{n} i^2 + n = \frac{n(n+1)(2n+1)}{6} + n = \Theta(n^3)$$

$(i^2-1+1)\cdot 1 = i^2$

$\sum_{i=1}^{n} 1 = (n-1+1)\cdot 1 = n$

for the mini cheat sheet:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

ii) Snippet 2:

Algorithm 2

```
for i = 1,...,n do
    k ← 1
    while k ≤ i^2 do
        f()
        k ← 2k
    f()
```

$k:\ 1, 2, 4, \dots, i^2$

$2^0, 2^1, 2^2, \dots, 2^{\log_2 i^2}$

$$\sum_{i=1}^{n}\left(\left(\sum_{j=0}^{\lfloor \log_2 i^2 \rfloor} 1\right) + 1\right) = n + \sum_{i=1}^{n}\left(\sum_{j=0}^{\lfloor \log_2(i^2) \rfloor} 1\right) = n + \sum_{i=1}^{n}\left(\log_2(i^2)\right)$$

(+1 ignored for rest)

$$= n + \sum_{i=1}^{n} 2\log(i)$$

$$= 2\log(1) + 2\log(2) + \dots + 2\log(n) + n$$

$$= 2\log(n!) + n = \Theta(n \log n)$$

$$\frac{n}{2}^{n/2} \le n! \le n^n$$

Exercise 2.1 *Induction.*

(a) Prove via mathematical induction that for all integers $n \geq 5$,

$$2^n > n^2 .$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

Base Case: $n=5$ $\quad 2^5 = 32 > 5^2 = 25$

I.H.: For some integer $k \geq 5$ $\quad 2^k > k^2$ holds

I.S.: $2^{k+1} = 2 \cdot 2^k \overset{\text{I.H.}}{>} 2 \cdot k^2 = k^2 + k^2$

$\geq k^2 + 5k$ $\quad | k \geq 5$

$= k^2 + 2k + 3k$

$\geq k^2 + 2k + 15$ $\quad | k \geq 5$

$> k^2 + 2k + 1$

$= (k+1)^2$

By the principle of ...

Let x be a real number. Prove via mathematical induction that for every positive integer n, we have

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

We use a standard convention $0! = 1$, so $\binom{n}{0} = \binom{n}{n} = 1$ for every positive integer n.

Hint: You can use the following fact without justification: for every $1 \le i \le n$,

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}.$$

To Show:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \quad , \quad \text{for every positive integer } n$$

Base Case:

$n=1$

$$(1+x)^1 = 1+x \,//$$

$$\sum_{i=0}^{1} \binom{1}{i} x^i = \binom{1}{0} \cdot x^0 + \binom{1}{1} \cdot x = 1 + x \,//$$

I.H.:

Assume that the property holds for some positive integer k:

$$(1+x)^k = \sum_{i=0}^{k} \binom{k}{i} x^i$$

$(k+1)$

I.S. :

$$(1+x)^{k+1} = (1+x)(1+x)^k$$

$$\overset{\text{I.H.}}{=} (1+x)\sum_{i=0}^{k}\binom{k}{i}x^i$$

$$= \left(\sum_{i=0}^{k}\binom{k}{i}x^i\right) + \left(x\cdot\sum_{i=0}^{k}\binom{k}{i}x^i\right)$$

$$x\cdot\left(\binom{k}{0}x^0+\binom{k}{1}x^1+\ldots+\binom{k}{k}x^k\right)$$

$$=\binom{k}{0}x^1+\binom{k}{1}x^2+\ldots+\binom{k}{k}x^{k+1}$$

$$= \left(\sum_{i=0}^{k}\binom{k}{i}x^i\right) + \left(\sum_{i=0}^{k}\binom{k}{i}x^{i+1}\right)$$

$$= \left(\sum_{i=0}^{k}\binom{k}{i}x^i\right) + \left(\sum_{i=1}^{k+1}\binom{k}{i-1}x^i\right)$$

$$= \binom{k}{0}\cdot x^0 + \left(\sum_{i=1}^{k}\binom{k}{i}x^i\right) + \left(\sum_{i=1}^{k}\binom{k}{i-1}x^i\right) + \binom{k}{k}\cdot x^{k+1}$$

$$= \binom{k}{0}\cdot x^0 + \sum_{i=1}^{k}\left(\binom{k}{i}x^i + \binom{k}{i-1}x^i\right) + \binom{k}{k}\cdot x^{k+1}$$

$$\overset{\text{hint}}{=} \binom{k}{0}\cdot x^0 + \sum_{i=1}^{k}\binom{k+1}{i}x^i + \binom{k}{k}\cdot x^{k+1}$$

Hint: You can use the following fact without justification: for every $1 \le i \le n$,

$$\binom{n}{i}+\binom{n}{i-1}=\binom{n+1}{i}.$$

$$= \binom{k+1}{0}x^0 + \sum_{i=1}^{k}\binom{k+1}{i}x^i + \binom{k+1}{k+1}x^{k+1}$$

$$\binom{k}{0} = 1 = \binom{k+1}{0}$$

$$\binom{k}{k} = 1 = \binom{k}{k}$$

$$= \sum_{i=0}^{k+1}\binom{k+1}{i}x^i \quad \square$$

By the principle of mathematical induction, the property is true for every positive integer n.

Exercise 2.5 *Asymptotic growth of* $\ln(n!)$.

Recall that the factorial of a positive integer n is defined as $n! = 1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n$. For the following functions n ranges over $\mathbb{N}_{\geq 2}$.

(a) Show that $\ln(n!) \leq O(n \ln n)$.

Hint: *You can use the fact that* $n! \leq n^n$ *for* $n \in \mathbb{N}_{\geq 2}$ *without proof.*

$$\ln(n!) \leq \ln(n^n) \quad | \text{ log is monoton + hint}$$

$$= n \ln(n) \quad \Rightarrow \quad \ln(n!) \leq O(n \ln n)$$

(b) Show that $n \ln n \leq O(\ln(n!))$.

Hint: *You can use the fact that* $\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n!$ *for* $n \in \mathbb{N}_{\geq 2}$ *without proof.*

$$\ln(n!) \geq \ln\left(\frac{n}{2}^{\frac{n}{2}}\right) \quad | \text{ log is monoton + hint}$$

$$= \frac{n}{2}(\ln n - \ln 2)$$

$$\Rightarrow \quad 2\ln(n!) \geq n\ln n - n\ln 2$$

$$\Rightarrow \quad 2\ln(n!) + \underbrace{n\ln 2} \geq n \ln n$$

$$\star: \quad n\ln 2 \leq \ln 2 + \sum_{i=1}^{n} \ln(i) = \ln 2 + (\ln(1) + \ln(2) + \dots + \ln(n))$$

$$= \ln 2 + \ln(n!) \leq 2\ln(n!)$$

$$\Rightarrow \quad n\ln n \leq 2\ln(n!) + n\ln 2 \overset{\star}{\leq} 4\ln(n!)$$

$$n \ln n \leq 4\ln(n!) \Rightarrow n\ln n \leq O(\ln n!)$$