Definition 1 (O-Notation)**.** For $f : N \to \mathbb{R}^+$,

constant $\quad 100n^2 \leq 100000 \cdot n^2$

$$O(f) := \{g : N \to \mathbb{R}^+ \mid \exists C > 0 \, \forall n \in N \; g(n) \leq C \cdot f(n)\}.$$

$$\mathcal{O}(n^2) = \{n, log(n), \sqrt{n}, log^2(n), 1, 100n^2, \cdots\}$$

Notation:

$g(n) \leq O(f(n))$

$g \leq O(f)$

"Functions g that fulfill the conditions are in $O(f)$ "

Definition 1 (O-Notation)**.** For $f : N \to \mathbb{R}^+$,

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Definition 1 (Ω-Notation)**.** For $f : N \to \mathbb{R}^+$,

$$\Omega(f) := \{g : N \to \mathbb{R}^+ \mid f \leq O(g)\}.$$

We write $g \geq \Omega(f)$ instead of $g \in \Omega(f)$.

Definition 2 (Θ-Notation)**.** For $f : N \to \mathbb{R}^+$,

$$\Theta(f) := \{g : N \to \mathbb{R}^+ \mid g \leq O(f) \text{ and } f \leq O(g)\}.$$

We write $g = \Theta(f)$ instead of $g \in \Theta(f)$.

In other words, for two functions $f, g : N \to \mathbb{R}^+$ we have

$$g \geq \Omega(f) \Leftrightarrow f \leq O(g)$$

and

$$g = \Theta(f) \Leftrightarrow g \leq O(f) \text{ and } f \leq O(g).$$

Theorem 1 (Theorem 1.1 from the script)**.** *Let N be an infinite subset of $\mathbb{N}$ and $f : N \to \mathbb{R}^+$ and $g : N \to \mathbb{R}^+$.*

- *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, *then* $f \leq O(g)$, *but* $f \neq \Theta(g)$.
- *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C \in \mathbb{R}^+$, *then* $f = \Theta(g)$.
- *If* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$, *then* $f \geq \Omega(g)$, *but* $f \neq \Theta(g)$.

FS 20

claim	true	false
$(2n+n^2+3)^2 = \Theta(n^4)$	☒	☐
$\frac{n}{\log n} \le \mathcal{O}(\sqrt{n})$ $\ge \Omega(\sqrt{n})$	☐	☒
$\log(n!) = \Theta(n \log n)$	☒	☐
$\sum_{i=1}^{\log_5 n} 5^i \ge \Omega(n \log n)$	☐	☒

Asymptotic Notation

Mini cheat-sheet

$\lim_{n\to\infty}$: $1 < \log(\log(n)) < \log(n) < \sqrt{n} < n < n\cdot\log(n) < n\cdot\sqrt{n} < n^2 < 2^n < n! < n^n$

$n^x < x^n$ (x being fixed)

Sums

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots n^2 = \frac{n(n+1)(2n+1)}{6}$$

Geometric series : $$\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}$$

$$\sum_{k=0}^{3} 3^k = 3^0+3^1+3^2+3^3 = \frac{3^4-1}{3-1} = 40$$

Factorial

$$\frac{n}{2}^{\frac{n}{2}} \le n! \le n^n$$

Θ : $n\cdot\ln n = \Theta(\ln(n!))$

$\Rightarrow n\cdot\ln n \le O(n!) \wedge n\cdot\ln n \ge \Omega(n!)$

$$(n^2+2n+3)^2 = n^4 + \ldots$$

$$n^4$$

$$\frac{\frac{n}{\log n}}{\sqrt{n}} = \frac{\sqrt{n}}{\log n} \to \infty$$

$$\log\left(\frac{n}{2}^{\frac{n}{2}}\right) \le \log(n!) \le \log(n^n)$$

$$= \frac{n}{2}\log\left(\frac{n}{2}\right)$$

$$= \frac{n}{2}(\log n - \log 2)$$

$$= \frac{n\log n}{2} - \frac{n\log 2}{2}$$

$$= n\log n\left(\frac{1}{2} - \frac{\log 2}{2\log n}\right)$$

$\to 0$

$$\log(n^n)$$

$$= n\log(n)$$

$$n\log(n) \le \log n! \le n\log n$$

$$\sum_{i=1}^{\log_5 n} 5^i = \frac{5^{\log_5 n + 1} - 1}{5-1} - 5^0$$

$$= \frac{5\cdot n - 1}{4} - 1 \rightsquigarrow n$$

$$n \ge \Omega(\cancel{n \log n})$$

was tested in old exams

Exercise 1.2 *Sums of powers of integers.*

$$\left\}\ \sum_{i=1}^{n} i^3 = \Theta(n^4)\right.$$

↳ add to mini cheat sheet

(a) Show that, for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \le n^4$.

(b) Show that for all $n \in \mathbb{N}_0$, we have $\sum_{i=1}^{n} i^3 \ge \frac{1}{2^4} \cdot n^4$.

Hint: *Consider the second half of the sum, i.e.,* $\sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3$. *How many terms are there in this sum? How small can they be?*

Together, these two inequalities show that $C_1 \cdot n^4 \le \sum_{i=1}^{n} i^3 \le C_2 \cdot n^4$, where $C_1 = \frac{1}{2^4}$ and $C_2 = 1$ are two constants independent of n. Hence, when n is large, $\sum_{i=1}^{n} i^3$ behaves "almost like n^4" up to a constant factor.

(c)* Show that parts (a) and (b) generalise to an arbitrary $k \ge 4$, i.e., show that $\sum_{i=1}^{n} i^k \le n^{k+1}$ and that $\sum_{i=1}^{n} i^k \ge \frac{1}{2^{k+1}} \cdot n^{k+1}$ holds for any $n \in \mathbb{N}_0$.

★ know this, add to mini cheat sheet

★ $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$

a) $\sum_{i=1}^{n} i^3 \le \sum_{i=1}^{n} n^3 = n \cdot n^3 = n^4$

b) $\sum_{i=1}^{n} i^3 \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^3 \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^3 = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right)\left(\frac{n}{2}\right)^3$

(*) $\ge \left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^3 = \frac{1}{2^4} \cdot n^4$

Floor/Ceiling functions:

$\lfloor 2.4 \rfloor = 2 \qquad \lceil 2.4 \rceil = 3$

gets rid of the fractional part in the way that we want

$$\left(n - \left\lceil \frac{n}{2} \right\rceil + 1 \ge \frac{n}{2} \quad \text{because} \quad \frac{n}{2} \ge \left\lceil \frac{n}{2} \right\rceil - 1 \quad \text{by def of } \lceil\ \rceil \right)$$

c) $\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1}$

$\sum_{i=1}^{n} i^k \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k = \left(n - \left\lceil \frac{n}{2} \right\rceil + 1\right) \cdot \left(\frac{n}{2}\right)^k$

$\ge \left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^k = \frac{1}{2^{k+1}} \cdot n^{k+1}$

Exercise 1.4 *Proving Inequalities.*

(a) Prove the following inequality by mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}\leq\frac{1}{\sqrt{3n+1}},\quad n\geq 1.$$

In your solution, you should address the base case, the induction hypothesis and the induction step.

(b)* Replace $3n+1$ by $3n$ on the right side, and try to prove the new inequality by induction. This inequality is even weaker, hence it must be true. However, the induction proof fails. Try to explain to yourself how is this possible?

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.

Base Case: $n=1$ $\quad \frac{1}{2} \leq \frac{1}{\sqrt{4}} = \frac{1}{2}$

I.H.: For some k $\quad \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k} \leq \frac{1}{\sqrt{3k+1}}$

I.S.: $\quad \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k}\cdot\frac{2(k+1)-1}{2(k+1)}$

$$\overset{\text{I.H.}}{\leq} \frac{1}{\sqrt{3k+1}}\cdot\frac{2(k+1)-1}{2(k+1)}$$

$$= \frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} \overset{?}{\leq} \frac{1}{\sqrt{3k+4}} = \frac{1}{\sqrt{3(k+1)+1}}$$

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k+1}}{\sqrt{3k+4}}$$

$$\Leftrightarrow \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k+1}{3k+4}$$

$$\Leftrightarrow (4k^2+4k+1)\cdot(3k+4) \leq (4k^2+8k+4)(3k+1)$$

$$\Leftrightarrow 12k^3+16k^2+12k^2+16k+3k+4 \leq 12k^3+4k^2+24k^2+8k+12k+4$$

$$\Leftrightarrow 28k^2+19k \leq 28k^2+20k$$

$$\Leftrightarrow 0 \leq k \qquad \text{true since } k\in\mathbb{N} \quad (1\ldots)$$

By the principle of ...

b)

Solution:

Sometimes it is easier to prove more than less. This simple approach does not work for the weaker inequality as we are using a weaker (and insufficiently so!) induction hypothesis in each step.

If we try to do the same proof as above, we need to show in the induction step that

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{3k}}{\sqrt{3k+3}}.$$

Continuing as above, we get that we want to show that

$$\begin{aligned}\frac{2k+1}{2k+2} \leq \sqrt{\frac{3k}{3k+3}} &\iff \left(\frac{2k+1}{2k+2}\right)^2 \leq \frac{3k}{3k+3} \\ &\iff (4k^2+4k+1)(3k+3) \leq (4k^2+8k+4)(3k) \\ &\iff 12k^3+24k^2+15k+3 \leq 12k^3+24k^2+12k \\ &\iff 3k+3 \leq 0,\end{aligned}$$

which is not true.

However, as argued above in the exercise statement, the inequality is still true. We are just not able to prove it directly via mathematical induction.