b) **(This subtask is from August 2019 exam).** Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by } 2;$$
$$T(1) = 4.$$

definition (can be applied everywhere)

Prove by mathematical induction that

$$\star \quad T(n) \geq 6n^2 - 2n$$

(property to prove)

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$.

(ind. over which variable ?)

Base Case: $(k=0)$ $n = 2^0 = 1$ $\quad T(2^0) = T(1) \overset{def}{=} 4 \geq 6 \cdot 1^2 - 2 \cdot 1 = 4$

I.H.: Property holds for $b = 2^a$ for some $a \in \mathbb{N}_0$

$$T(2^a) \geq 6 \cdot (2^a)^2 - 2 \cdot (2^a)$$

I.S.: $(2^{a+1} = 2 \cdot b)$

$$\begin{aligned}
T(2^{a+1}) &\overset{def}{\geq} 4 \cdot T(2^a) + 3 \cdot 2^{a+1} \\
&\overset{I.H.}{\geq} 4 \cdot (6 \cdot (2^a)^2 - 2 \cdot 2^a) + 3 \cdot 2^{a+1} \\
&= 24 \cdot 2^{2a} - 4 \cdot 2^{a+1} + 3 \cdot 2^{a+1} \\
&= 24 \cdot 2^{2a} - 2^{a+1} \\
&= 6 \cdot 4 \cdot 2^{2a} - 2^{a+1} \\
&= 6 \cdot 2^{2a+2} - 2^{a+1} \\
&= 6 \cdot (2^{a+1})^2 - 2^{a+1} \\
&\geq 6 \cdot (2^{a+1})^2 - 2 \cdot (2^{a+1})
\end{aligned}$$

$| \; 4 \cdot (-2 \cdot 2^a) = -8 \cdot 2^a = -4 \cdot 2 \cdot 2^a = -4 \cdot 2^{a+1}$

$| \; 4 \cdot 2^{2a} = 2^2 \cdot 2^{2a} = 2^{2a+2}$

$| \; 2^{2a+2} = 2^{2(a+1)} = (2^{a+1})^2$

last step explanation: If we know y is ≥ 0, then this implies $x - y \geq x - 2y$

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/ 3 P b) *Induction:* Consider the sequence $\{L_n\}_{n\geq 1}$ defined via $L_1 = L_2 = 1$ and the recurrence $L_{n+1} = L_n + 2L_{n-1}$ for $n \geq 2$.

Show that for all $n \in \mathbb{N} \setminus \{0\}$, the following equation holds:

$$\sum_{i=1}^{n} 2^{n-i} \cdot L_i^2 = L_n L_{n+1}.$$

Base Case: $(n=1)$ $\sum_{i=1}^{1} 2^{1-i} \cdot L_i^2 = 2^0 \cdot L_1^2 = 1 \cdot L_1^2 = 1 = 1 \cdot 1 = L_1 \cdot L_2$

I.H.: $k \in \mathbb{N} \setminus \{0\}$. Assume that $\sum_{i=1}^{k} 2^{k-i} \cdot L_i^2 = 2^{k-1} \cdot L_1^2 + 2^{k-2} \cdot L_2^2 + \ldots + L_k^2 = L_k L_{k+1}$

★ Expand the sum if you get stuck ♡

I.S.: $2^k L_1^2 + 2^{k-1} L_2^2 + \ldots + 2 L_k^2 + L_{k+1}^2$

$= 2 (2^{k-1} L_1^2 + 2^{k-2} L_2^2 + \ldots + L_k^2) + L_{k+1}^2$

$\overset{\text{I.H.}}{=} 2 (L_k L_{k+1}) + L_{k+1}^2$

$= L_{k+1} (2L_k + L_{k+1})$

$\overset{\text{def}}{=} L_{k+1} L_{k+2}$