Mini-exam (Induction + Asymptotic Notation)

FS23

/ 5 P

a) *Asymptotic notation quiz:* For each of the following claims, state whether it is true or false. You get 1P for a correct answer, -1P for a wrong answer, 0P for a missing answer. You get at least 0 points in total.

Assume $n \geq 4$.

	Claim	true	false
$\lim_{n\to\infty} \frac{n^3+n^4}{n^4} = \lim_{n\to\infty} \frac{\cancel{n^4}(\overset{\to 0}{\frac{1}{n}}+1)}{\cancel{n^4}} = 1$	$n^3 + n^4 = \Theta(n^4)$	☒	☐
$n^x > \log(n)^{100}$ $\lim_{n\to\infty} \frac{n^{10}}{\log n^{100}} = \frac{10n^9}{100 \log n^{99} \cdot \frac{1}{n}}$ 1 more L'hopital	$n^{10} \leq O(\log(n)^{100})$	☐	☒
$n! > \frac{n}{2}^{\frac{n}{2}} > 2^n$	$1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \leq O(2^n)$	☐	☒
	Suppose $a_1 = 1$ and $a_{i+1} = 3a_i + 1$ for all $i \geq 2$. Then $a_n \leq O(4^n)$.	☒	☐
$\frac{2^{10\log n}}{2^{20\log n}} = 2^{10\log n - 20\log n} = 2^{-10\log n} = \frac{1}{2^{10\log n}} \xrightarrow{n\to\infty} 0$	$2^{10\log(n)} = \Theta(2^{20\log(n)})$ $\leq O(2^{20\log(n)})$	☐	☒

$a_1 = 1 = 1$
$a_2 = 4 = 3^1 + 3^0$
$a_3 = 13 = 3^2 + 3^1 + 3^0$
$a_4 = 40 = 3^3 + 3^2 + 3^1 + 3^0$

↳ geometric series

$a_n = \sum_{k=0}^{n-1} 3^k = \frac{1-3^n}{1-3} = \Theta(3^n)$
$\Rightarrow \leq O(4^n)$

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b) *Induction:* Consider the Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$, which are given by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Show by mathematical induction that for any integer $n \geq 0$, HS21

$$F_{n+1}^2 \geq \sum_{k=0}^{n} F_k^2.$$

Hint: *Use the facts that $F_{n+1} \geq F_n$ and $F_n \geq 0$ for all $n \in \mathbb{N}$ (you don't need to justify that).*

Base Case: $n=0$ $\quad F_1^2 = 1 \geq \sum_{k=0}^{0} F_k^2 = F_0^2 = 0$

$n=1$ $\quad F_2^2 \overset{\text{def}}{=} 1 \geq \sum_{k=0}^{1} F_k^2 = F_0^2 + F_1^2 = 0+1 = 1$

I.H.: For som $m \geq 1$ $\quad F_{m+1}^2 \geq \sum_{k=0}^{m} F_k^2$

I.S.: $m \to m+1$:

$$\begin{aligned}
F_{m+1+1}^2 &\overset{\text{def}}{=} (F_{m+1} + F_m)^2 \\
&= F_{m+1}^2 + 2F_{m+1}F_m + F_m^2 \\
&\geq \sum_{k=0}^{m} F_k^2 + 2F_{m+1}F_m + F_m^2 \\
&\geq \sum_{k=0}^{m} F_k^2 + 2F_{m+1}F_m \qquad | F_m \geq 0,\ F_m^2 \geq 0 \\
&= \sum_{k=0}^{m} F_k^2 + F_{m+1}(F_m + F_m) \\
&\geq \sum_{k=0}^{m} F_k^2 + \underbrace{F_{m+1}(F_m + F_{m-1})}_{=F_{m+1}^2}. \qquad | F_m \geq F_{m-1} \\
&\geq \sum_{k=0}^{m+1} F_k^2
\end{aligned}$$

(By the principle ...)